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Denisov's theorem on recurrence coefficients

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Abstract

Recently Denisov (aka Dennisov) (Proc. Amer. Math. Soc.) has proved the following remarkable extension of Rakhmanov's theorem (Math. USSR-Sb. 46 (1983) 105; Russian Original, Mat. Sb. 118 (1982) 104) (see also (Mate et al., Constr. Approx., 1 (1985) 63; Nevai, J. Approx. Theory 65 (1991) 322)) which was conjectured in (Nevai, in: Approximation Theory IV, Vol. II, Academic Press, New York, 1989, pp. 449–489, Conjecture 2.7, p. 453).

Theorem 1 (Denisov). *If the measure* μ *defined on* $\mathbf{R}$ *has compact support*, $\mu' > 0$ *almost everywhere in* $[-1, 1]$, *and the set* $\operatorname{supp}(\mu)\backslash[-1-\varepsilon, 1+\varepsilon]$ *is finite for every* $\varepsilon > 0$, *then*

$$\lim_{n\to\infty} a_n(\mu) = \tfrac{1}{2} \tag{1}$$

and

$$\lim_{n\to\infty} b_n(\mu) = 0. \tag{2}$$

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Here $a_n(\mu)$ and $b_n(\mu)$ denote the recurrence coefficients for the orthonormal polynomials $\{p_n(\mu)\}$ associated with μ (here $p_n(\mu, x) = \gamma_n x^n + \cdots$). Rakhmanov's theorem [10] asserts (1) and (2) under the hypothesis that μ is supported in $[-1, 1]$ and $\mu' > 0$ almost everywhere there. The significance of Denisov's extension lies in the fact that, according to Blumenthal's theorem [1], the limit relations in (1) and (2) imply that $[-1, 1] \subseteq \mathrm{supp}(\mu)$ and $\mathrm{supp}(\mu) \backslash [-1-\varepsilon, 1+\varepsilon]$ is finite for every $\varepsilon > 0$.[2]

Denisov used operator theoretic arguments. In view of the importance of the result, it is worthwhile to give another proof which is based on standard and purely "orthogonal polynomial" techniques which we have developed in a series of papers in the 1980's when extending Szegő's theory (cf. [4–6]).

Proof. Just like Denisov, we also use the fact that if the support of a measure σ lies in $[-1, 1]$ and $\sigma' > 0$ on a set of (Lebesgue) measure at least $2 - \delta$, then

$$\limsup_{n \to \infty} \left(|a_n(\sigma) - 1/2| + |b_n(\sigma)| \right) \leqslant \theta(\delta), \tag{3}$$

where $\theta(\delta)$ depends only on δ, and $\theta(\delta) \to 0$ as $\delta \to 0$. Denisov proves this in detail in [3], but there is a fairly short proof using some well-known inequalities. In fact, let $v(t)$, $t \in [-\pi, \pi]$ be the measure that we obtain by projecting σ up the unit circle, and let $\Phi_n(v)$ be the recurrence coefficients (frequently called reflection or Verblunsky coefficients) for the Szegő type orthogonal polynomials with respect to v. It is well-known (see, e.g., [7, Lemma 7.8, p. 189] and [15, (3.14) and (3.15)]) that

$$a_k = \tfrac{1}{2} \sqrt{(1 - \Phi_{2k})(1 - \Phi_{2k-1}^2)(1 + \Phi_{2k-2})}$$

and

$$b_k = \tfrac{1}{2} [\Phi_{2k-1}(1 - \Phi_{2k}) - \Phi_{2k+1}(1 + \Phi_{2k})].$$

Therefore, it is sufficient to show that if $v' > 0$ on a set of measure at least $2\pi - \delta^*$, then

$$\limsup_{n \to \infty} |\Phi_n(v)| \leqslant \delta^*. \tag{4}$$

The latter follows from the inequality

$$|\Phi_{n+1}|^2 \leqslant \frac{2}{\pi} \int_0^{2\pi} \left(|q_n(e^{it})| \sqrt{v'(t)} - 1 \right)^2 dt + \frac{2}{\pi} \int_0^{2\pi} |q_n(e^{it})|^2 \, dv_s(t), \tag{5}$$

where v_s denotes the singular part (with respect to Lebesgue measure) of v and where q_n is an arbitrary algebraic polynomial of degree at most n (see [11, Lemmas 2 and 3] or [9, Theorem 15]). By the Fejér–Riesz representation theorem, every nonnegative

[2] However, it does not follow from (1) and (2) that $\mu' > 0$ almost everywhere in $[-1, 1]$. For instance, μ can be singular with respect to Lebesgue measure (see, e.g., [2,14]).

trigonometric polynomial can be written as $|p(e^{it})|^2$ with some algebraic polynomial p. Hence, inequality (5) implies

$$\limsup_{n\to\infty} |\Phi_n|^2 \leqslant \inf_T \left(\int_0^{2\pi} (\sqrt{T(t)}\sqrt{v'(t)} - 1)^2\, dt + \int_0^{2\pi} T(t)\, dv_s(t) \right), \tag{6}$$

where the infimum on the right is taken for all nonnegative trigonometric polynomials T. Since every nonnegative upper semi-continuous 2π-periodic function is a decreasing limit of 2π-periodic continuous functions and hence of nonnegative trigonometric polynomials, the preceding inequality remains valid (use the monotone convergence theorem) if the infimum in it is taken for all nonnegative upper semi-continuous functions T. By Lusin's theorem [12, Theorem 2.24] there is a compact set $E \subseteq [0, 2\pi]$ of measure at least $2\pi - \delta^*$ such that v' is positive and continuous on E, and v_s is supported outside E. Therefore, in (6) we can set T equal to $1/v'$ on E and 0 outside E to conclude the proof of (4).

Having verified (3), we return now to the proof of the theorem. Inequality (3) implies that if β is a measure supported on $[-1-\varepsilon, 1+\varepsilon]$ and $\beta' > 0$ almost everywhere in $[-1, 1]$, then

$$\limsup_{n\to\infty} (|a_n(\beta) - (1+\varepsilon)/2| + |b_n(\beta)|) \leqslant \theta(2\varepsilon). \tag{7}$$

Let $\varepsilon > 0$ be fixed, and let

$$\mu - \mu_{|[-1-\varepsilon,1+\varepsilon]} = \sum_{i=1}^{r} c_i \delta_{x_i}, \qquad c_i > 0.$$

It is well-known that each x_i attracts a zero of $p_n(\mu)$ (see, e.g., [13, Theorem 6.1.1, p. 111]),[3] so that for sufficiently large n there are distinct zeros $x_{n,i}$, $1 \leqslant i \leqslant r$ of $p_n(\mu)$ with $x_{n,i} \to x_i$ as $n \to \infty$. Thus, if we set $q_r(x) = \prod_1^r (x - x_i)$ and $q_{n,r}(x) = \prod_1^r (x - x_{n,i})$, then uniformly in $x \in [-1-\varepsilon, 1+\varepsilon]$ we have

$$\left| \frac{q_r(x)}{q_{n,r}(x)} \right| \leqslant 1 + \tau_n, \quad \text{where } \tau_n \to 0 \quad \text{as } n \to \infty. \tag{8}$$

Next, set $d\beta(x) = q_r^2(x)\, d\mu(x)$, and compare the leading coefficients of the corresponding orthogonal polynomials $\gamma_n(\mu)$ with $\gamma_{n-r}(\beta)$. Note that the support of β lies in $[-1-\varepsilon, 1+\varepsilon]$ and $\beta' > 0$ almost everywhere in $[-1, 1]$, so that for it inequality (7) holds. Making use the extremal property of orthogonal polynomials we obtain

$$\frac{1}{\gamma_n^2(\mu)} = \min_{P_n(x) = x^n + \dots} \int_{\mathbf{R}} P_n^2\, d\mu \leqslant \min_{P_{n-r}(x) = x^{n-r} + \dots} \int_{\mathbf{R}} P_{n-r}^2 q_r^2\, d\mu = \frac{1}{\gamma_{n-r}(\beta)^2} \tag{9}$$

[3] In fact, each x_i attracts precisely one zero of each $p_n(\mu)$.

and, similarly,

$$\frac{1}{\gamma_{n-r}^2(\beta)} = \min_{P_{n-r}(x)=x^{n-r}+\dots} \int_{-1-\varepsilon}^{1+\varepsilon} P_{n-r}^2 q_r^2\, d\mu$$
$$\leqslant (1+\tau_n)^2 \min_{P_{n-r}(x)=x^{n-r}+\dots} \int_{-1-\varepsilon}^{1+\varepsilon} P_{n-r}^2 q_{n,r}^2\, d\mu$$
$$\leqslant (1+\tau_n)^2 \int_{\mathbf{R}} \left(\frac{p_n(\mu)}{\gamma_n(\mu) q_{n,r}}\right)^2 q_{n,r}^2\, d\mu = \frac{(1+\tau_n)^2}{\gamma_n^2(\mu)}.$$

Therefore,

$$\gamma_{n-r}(\beta) \leqslant \gamma_n(\mu) \leqslant (1+\tau_n)\gamma_{n-r}(\beta). \tag{10}$$

Now this and the same inequality for $n+1$ instead of n yield for $a_n(\mu) = \gamma_n(\mu)/\gamma_{n+1}(\mu)$ the estimate

$$\frac{1}{1+\tau_{n+1}} a_{n-r}(\beta) \leqslant a_n(\mu) \leqslant (1+\tau_n) a_{n-r}(\beta), \tag{11}$$

from which we obtain

$$\limsup_{n\to\infty} a_n(\mu) = \limsup_{n\to\infty} a_n(\beta), \quad \liminf_{n\to\infty} a_n(\mu) = \liminf_{n\to\infty} a_n(\beta). \tag{12}$$

Since $\varepsilon > 0$ was arbitrary, formula (1) is a consequence of (12) and (7).

Next, consider the polynomials

$$S_{n,\pm}(x) = \frac{1}{2}\left(p_n(\mu, x) \pm \frac{\gamma_n(\mu) p_{n-r}(\beta, x) q_r(x)}{\gamma_{n-r}(\beta)}\right).$$

By orthogonality,

$$\int_{\mathbf{R}} (S_{n,-})^2\, d\mu + \int_{\mathbf{R}} (S_{n,+})^2 d\mu = \frac{1}{2} + \frac{\gamma_n^2(\mu)}{2\gamma_{n-r}^2(\beta)} \leqslant \frac{1+(1+\tau_n)^2}{2},$$

where we used (10). $S_{n,+}$ is a polynomial of degree n with leading coefficient $\gamma_n(\mu)$ so that by the extremal property (9), the second integral on the left is at least 1. Therefore, it follows that

$$\int_{\mathbf{R}} (S_{n,-})^2 \mu \leqslant \tau_n + \tau_n^2/2. \tag{13}$$

We have for all measures α

$$b_n(\alpha) = \int_{\mathbf{R}} x p_n^2(\alpha, x)\, d\alpha(x).$$

Using this formula for both $b_n(\mu)$ and $b_n(\beta)$, we obtain

$$\left| b_n(\mu) - \frac{b_{n-r}(\beta)\gamma_n^2(\mu)}{\gamma_{n-r}^2(\beta)} \right| = \left| \int_{\mathbf{R}} x \left[p_n^2(\mu, x) - \frac{\gamma_n^2(\mu)}{\gamma_{n-r}^2(\beta)} p_{n-r}^2(\beta, x) q_r^2(x) \right] d\mu(x) \right|$$
$$\leqslant C \int_{\mathbf{R}} |S_{n,-} S_{n,+}| \, d\mu \leqslant C \left(\int_{\mathbf{R}} (S_{n,-})^2 \, d\mu \right)^{1/2} \left(\int_{\mathbf{R}} (S_{n,+})^2 \, d\mu \right)^{1/2}$$

where the constant C depends only on the location of the support of μ. We can write

$$\left(\int_{\mathbf{R}} (S_{n,+})^2 \, d\mu \right)^{1/2}$$
$$\leqslant \frac{1}{2} \left(\int_{\mathbf{R}} p_n^2(\mu) \, d\mu \right)^{1/2} + \frac{\gamma_n(\mu)}{2\gamma_{n-r}(\beta)} \left(\int_{\mathbf{R}} p_{n-r}^2(\beta) q_r^2 \, d\mu \right)^{1/2}$$
$$= \frac{1}{2} \left(1 + \frac{\gamma_n(\mu)}{\gamma_{n-r}(\beta)} \right) \leqslant 1 + \tau_n/2$$

(see (10)), so that we obtain from (13) and (8)

$$\lim_{n \to \infty} \left| b_n(\mu) - \frac{b_{n-r}(\beta)\gamma_n^2(\mu)}{\gamma_{n-r}^2(\beta)} \right| = 0.$$

Therefore, using again (10), we see that

$$\limsup_{n \to \infty} |b_n(\mu)| = \limsup_{n \to \infty} |b_{n-r}(\beta)|.$$

Now formula (2) follows from here, inequality (7), and from the fact that $\varepsilon > 0$ in (7) is arbitrary. □

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